Technical Notes

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Corrected Formulas for Natural Frequencies of Cantilever Beams Under Uniform Axial Tension

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I. Introduction

NATURAL frequencies of beams under axial tensile force are of interest in many applications such as rotating fan blades, turbine blades, etc. There have been extensive studies on beam dynamics considering the effects of rotation. Rosales and Filipich [1], for example, studied the dynamic stability of a spinning beam with an axial dead load. Naguleswaran [2] presented a study of lateral vibration of a centrifugally tensioned uniform Euler–Bernoulli beam. He developed the database of nondimensional parameters of speed and natural frequencies for various offset ratios of a beam. Lee [3] studied the dynamic response of a rotating Timoshenko beam subjected to axial forces and a moving load.

Closed-form solutions for the natural frequencies of rotating pinned–pinned beams are available (for example, Timoshenko et al. [4]). However, other boundary conditions are not readily amenable to such a treatment. Gorman [5] presented the variation of the frequency parameter with axial tension parameter for the six modes of clamped–pinned and clamped–clamped beams. Bokaian [6] presented a comprehensive study of the effect of a constant axial tensile force on the natural frequencies and mode shapes of a uniform single-span beam, with different combinations of the end conditions. Although simple relations could capture the behavior of beams with various boundary conditions, clamped–free beams showed deviant behavior. Thus, the present study focuses on the natural frequencies of a clamped–free beam under uniform axial tension.

The relations available for finding the fundamental frequency of a cantilever beam under uniform axial tension are summarized next:

The Bokaian [6] characteristic equation is

$$\begin{aligned}\Omega^2 &+ \Omega U \sinh(U + \sqrt{U^2+\Omega^2})^{1/2} \times \sin(-U + \sqrt{U^2+\Omega^2})^{1/2} \\ &+ (2U^2+\Omega^2)\cosh(U+\sqrt{U^2+\Omega^2})^{1/2} \\ &\times \cos(-U+\sqrt{U^2+\Omega^2})^{1/2} = 0 \end{aligned} \tag{1}$$

where $\Omega = \omega L^2 \sqrt{\rho A/EI}$ and $U = PL^2/2EI$

The Bokaian [6] lower bound is

$$\omega_{a1} = \pi/2L\sqrt{P/\rho A} \tag{2}$$

for $PL^2/EI > 24$.

The Bokaian [6] upper bound is

$$\omega_{a1} = \omega_1\sqrt{1 + 0.375PL^2/EI} \tag{3}$$

for $PL^2/EI > 24$.

The handbook formula based on shape function given in Eq. (5) [7] is

$$\omega_{a1} = \omega_1\sqrt{1 + 0.357(PL^2/EI)} \tag{4}$$

where ω_{a1} and ω_1 are the fundamental frequencies of the beam with and without axial force in rad/s.

The variation of the frequency parameter $(\omega_{a1}/\omega_1)^2$ with tension parameter PL^2/EI for various preceding formulas can be observed from Fig. 1. Lower and upper bounds on natural frequency by Bokaian [6] are represented by Bokaian-L and Bokaian-U, respectively, and numerical solutions of the characteristic equation given by Bokaian is represented by Bokaian-trans. For comparison purposes, the finite element solution obtained using a 2-D elastic beam element with the stress-stiffening option enabled (in ANSYS [8]) was also given.

It can be observed from Fig. 1 that finite element results and results generated from Bokaian's [6] characteristic equation are perfectly matching over the entire range, whereas the handbook formula gives close results up to $PL^2/EI \approx 5$. The lower and upper bounds of Bokaian also do not provide satisfactory estimates over a wide range. Also the transcendental characteristic (1) has to be typically solved numerically to obtain Ω for a given value of U. Thus, there is a need for easy-to-use yet accurate relations for fundamental frequency. Fundamental frequency relation was derived in this work using the Rayleigh quotient with a *revised shape function* and relations for higher frequencies are obtained using Rayleigh–Ritz method. Simple, easy-to-use, and accurate expressions for the first four natural frequencies are provided.

II. Proposed Shape Function

It is standard practice to use static deflection curve under self-weight as a shape function, as given next [9]:

$$v = \frac{\rho A g}{2EI}\left[\frac{Lx^3}{3} - \frac{L^2x^2}{2} - \frac{x^4}{12}\right] \tag{5}$$

Using the energy method, the fundamental frequency can be obtained using the Rayleigh quotient [10]. It is proposed here that the fundamental frequency results can be estimated better using a shape representing the static deflection of a cantilever beam under self-weight *in the presence of axial tension*. The governing differential equation for static deflection of a cantilever beam under self-weight in the presence of axial tension is given by

$$EI\frac{d^2v}{dx^2} = \frac{q}{2}(L-x)^2 - P(v_L - v) \tag{6}$$

subject to the boundary conditions

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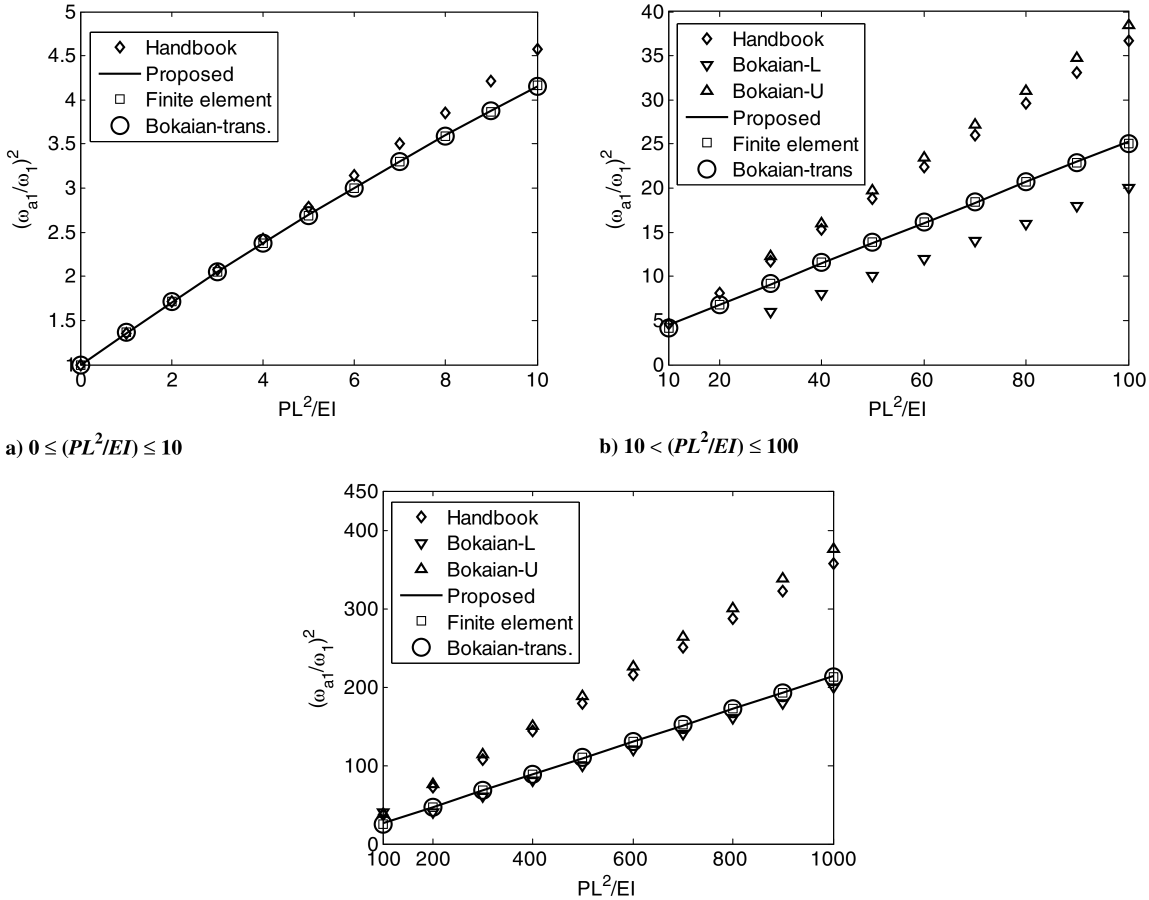


a) $0 \leq (PL^2/EI) \leq 10$

b) $10 < (PL^2/EI) \leq 100$

c) $(PL^2/EI) > 100$

Fig. 1 Comparison of estimates of the fundamental frequency (Hz) of a cantilever beam under axial force.

$$v(0) = 0, \qquad \frac{\mathrm{d}v(0)}{\mathrm{d}x} = 0 \tag{7}$$

where v_L is the static deflection at the free end of the cantilever beam and q is the self-weight per unit length.

Upon solution, we get

$$v(x) = -C_1 e^{\sqrt{\frac{P}{EI}}x} - C_2 e^{-\sqrt{\frac{P}{EI}}x} - \frac{qL}{P}x + \frac{q}{2P}x^2 - v_L + \frac{qL^2}{2P} + \frac{EIq}{P^2} \tag{8}$$

where

$$\begin{aligned} C_1 &= \frac{qL^2}{4P} + \frac{EIq}{2P^2} - \frac{qL\sqrt{EI}}{2P^{3/2}} - \frac{v_L}{2} \\ C_2 &= \frac{qL^2}{4P} + \frac{EIq}{2P^2} + \frac{qL\sqrt{EI}}{2P^{3/2}} - \frac{v_L}{2} \\ v_L &= \left(\frac{qL^2}{2P} + \frac{EIq}{P^2}\right) + \frac{qL\sqrt{EI}}{P^{3/2}}\left(\frac{e^{-\sqrt{P/EI}} - e^{\sqrt{P/EI}}}{e^{\sqrt{P/EI}} + e^{-\sqrt{P/EI}}}\right) \\ &\quad - \frac{qEI}{P^2}\left(\frac{1}{e^{\sqrt{P/EI}} + e^{-\sqrt{P/EI}}}\right) \end{aligned} \tag{9}$$

Static deflection expression in the presence of axial tension [Eq. (8)] was used to estimate the Rayleigh quotient in the present work. The computations were carried out using the symbolic processor in MATLAB [11]. A least-squares (polynomial) curve-fitting strategy was used to develop a simple formula for the fundamental frequency parameter.

III. Estimation of Higher Frequencies

For large values of tension parameter and for the lower modes, Bokaian [6] presented the lower and upper bounds for the ith natural frequency as

$$\omega_{al_i} = (\pi/2L)(2i-1)\sqrt{\frac{P}{\rho A}} \tag{10}$$

and

$$\omega_{au_i} = \omega_i\sqrt{1 + \frac{0.375}{(2i-1)^2}\frac{PL^2}{EI}} \tag{11}$$

respectively. In the present work, the higher frequencies of the cantilever beam under axial tension were obtained using the Rayleigh–Ritz method. A twelfth-order polynomial shape function satisfying the essential boundary conditions was used for this purpose. The computations were carried out using the symbolic processor in MATLAB.

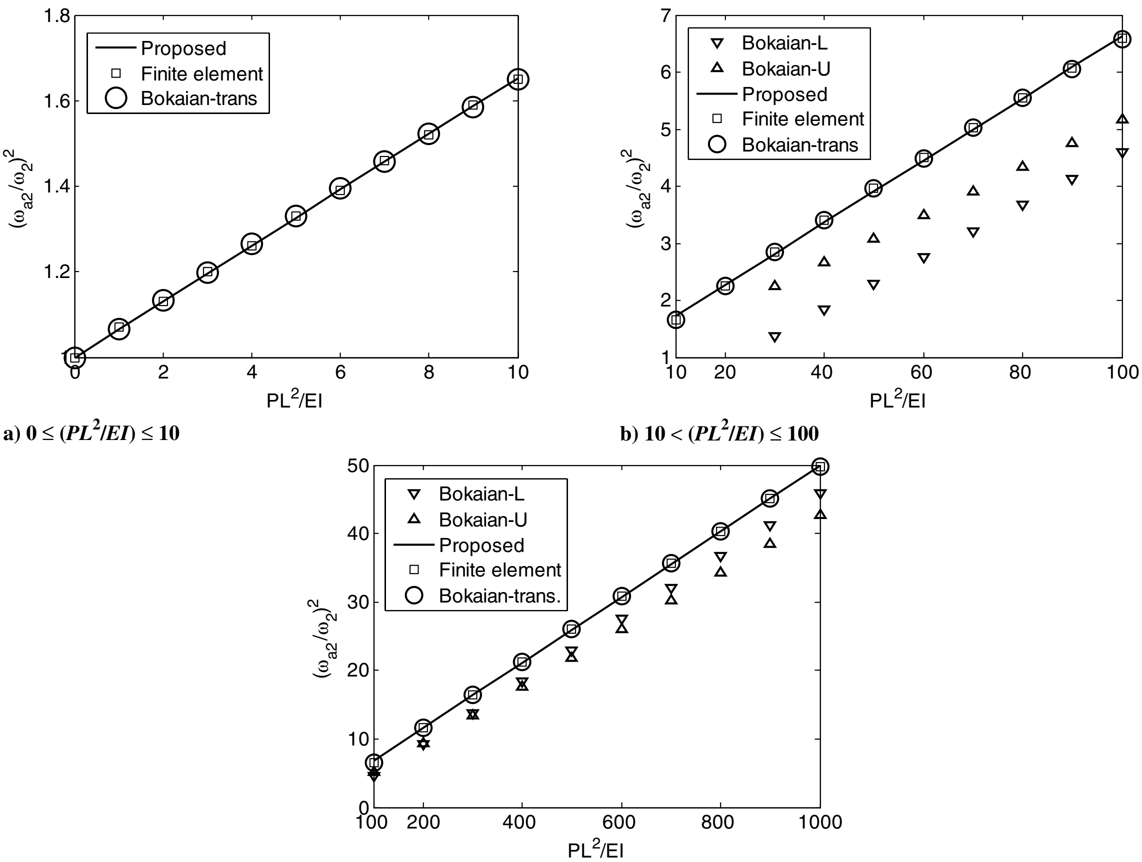
IV. Results and Discussion

After studying the natural frequency variation with tension parameter over a wide range, it is observed that it would be difficult to get a single linear relation that is valid over the entire range. Hence, the tension parameter is divided into three zones: 0–10, 10–100, and greater than 100. To get simple analytical expressions for the natural frequencies, Rayleigh–Ritz results are curve-fitted using MATLAB. The natural frequency relations for the first four modes can be given as follows:

Table 1 Natural frequencies of a cantilever beam under axial tension

Mode	Zone-1 $0 \leq (PL^2/EI) \leq 10$				Zone-2 $10 < (PL^2/EI) \leq 100$				Zone-3 $(PL^2/EI) > 100$			
	α	β	γ	Max % error[a]	α	β	γ	Max % error	α	β	γ	Max % error
1	1	0.3630	−0.0048	0.81	2.1977	0.2309	——	1.63	5.026	0.2088	——	1.80
2	1	0.0653	——	0.66	1.1868	0.0544	——	1.58	2.0055	0.0479	——	1.76
3	1	0.0203	——	0.50	1.0251	0.0191	——	1.16	1.3143	0.0170	——	1.47
4	1	0.0098	——	0.42	1.005	0.0096	——	1.12	1.1366	0.0087	——	1.42

[a]Max % error of ω_{a_i}/ω_i in proposed formulas to finite element solutions (ANSYS).



a) $0 \leq (PL^2/EI) \leq 10$

b) $10 < (PL^2/EI) \leq 100$

c) $(PL^2/EI) > 100$

Fig. 2 Comparison of estimates of second natural frequency (Hz) of cantilever beam with axial tension.

$$\left(\frac{\omega_a}{\omega}\right)^2 = \alpha + \beta\left(\frac{PL^2}{EI}\right) + \gamma\left(\frac{PL^2}{EI}\right)^2 \tag{12}$$

where α, β, and γ are given in Table 1 for various modes.

For the first two modes, the variation of the frequency parameter with tension parameter is illustrated in Figs. 1 and 2. Estimates from proposed formulas are compared with the lower and upper bounds of Bokaian and the finite element solution from ANSYS. It is observed that the proposed relations tally very well with the finite element solution obtained from ANSYS, as well as the roots of the Bokaian-transcendental equation for all of the modes. However, the approximate lower and upper bounds given by Bokaian are not satisfactory and, in fact (for the system studied here), the upper bound for higher modes (for example mode 2) no longer remains an upper-bound estimate. It was observed that the change in mode shapes is significant at low values of PL^2/EI but at higher values, the mode shapes do not change much.

V. Conclusions

Using a modified shape function, the available expression for the fundamental frequency of a cantilever beam with a tensile load at the free end was corrected to very high accuracy. Simple, easy-to-use, yet accurate expressions for the natural frequency parameter for the first four modes are developed for a wide range of tension parameter values. The accuracy of the proposed formulas was demonstrated against available results and a finite element solution from commercial software. Proposed relations give considerable improvement over existing formulas and tally excellently with the detailed finite element model for all of the modes.

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